Dynamical random graphs with memory

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We study the large-time dynamics of a Markov process whose states are finite but unbounded graphs. The number of vertices is described by a supercritical branching process, and the edges follow a certain mean-field dynamics determined by the rates of appending and deleting: the older an edge is, the lesser is the probability that it is still in the graph. The lifetime of any edge is distributed exponentially. We call its mean value (common for all edges) a parameter of memory, since it shows for how long the system keeps a particular connection between the vertices in the graph. We show that our model provides a bridge between two well-known models: when the parameter of memory goes to infinity this is a generalized model of random growth, and when this parameter is zero, i.e., no memory, our model behaves as a random graph. Thus by introducing a general class of dynamical graphs we have a unified overview on rather different models and the relations between them. We find all the critical values of the parameters at which our model exhibits phase transitions and describe the properties of the phase diagram. Finally, we compare and discuss the efficiency of the corresponding networks.

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I. INTRODUCTION

Randomly grown networks became a subject of intensive study over the last few years (see, e.g., Refs. [1–6] and the references therein). Examples of such systems range from the artificial structures such as the World Wide Web to the social networks and the biological neural networks. Clearly, the mathematical models describing these structures possess different properties. The authors of "Are randomly grown graphs really random?" [3] give a negative answer to this question, showing dramatic difference in the behavior of two classes of models.

We investigate a general model that apparently provides a bridge between the randomly grown graphs [3] and the classical random graphs [7,8]. Recall a definition of a randomly grown network according to Ref. [3]. Starting with a single vertex the graph evolves in a discrete time, acquiring a new vertex at each step. Also, at each step a new edge appears with a probability δ between two vertices chosen uniformly over all the existing vertices of the current graph. This is a simple but nontrivial example of a nonhomogeneous network where the degree of a vertex depends on its age. One of the basic features of this model is the monotonicity associated with the acquisition of edges: once an edge is introduced into a model it stays there forever.

On the other hand, there are many real world networks affected by the aging of the connections as well as nodes, e.g., social networks [6] lose permanently some connections, the neural networks change their architecture due to synaptic plasticity, or the so-called decaying networks [4] are subject to a permanent damage of the edges.

Our aim here is to analyze the concurrent roles of growth and aging resulting in removal of links, in the structure of a dynamical random graph. Let us introduce the model we study here.

A. The model

Let $V(t)$ and $\mathcal{L}(t)$ denote the set of vertices and the set of edges, correspondingly, at time t. We assume that at $t=0$ we have just one vertex with no edge: $|V(0)|=1$ and $\mathcal{L}(0)$ is an empty set.

With time, our graph accumulates vertices, meaning that $|V(t)|$ is increasing, in the following random way. Every vertex in the graph generates new vertices with intensity γ and independent of the rest of the system. Put in another way, with every vertex in the graph we associate a Poisson process with intensity γ, every occurrence of which corresponds to the appearance of a new vertex in the graph. Each newly acquired vertex generates new ones in the same fashion and independent of the rest. Thus the number of the vertices $|V(t)|$ is described by a supercritical branching process for more details, see Ref. [9]). In particular, it is easy to see that the averaged size of our network grows exponentially:

$$E|V(t)|=e^{\gamma t}. \tag{1}$$

where E denotes the sign of mathematical expectation. As soon as there are two vertices in the graph, from every vertex the edges are drawn with a common intensity λ and independent of other processes. The end of an edge drawn at time t from a vertex $v \in V(t)$ is chosen uniformly among the rest of existing vertices $V(t)\setminus\{v\}$. In other words, with every vertex we associate another Poisson process with intensity λ, whose every occurrence corresponds to the appearance of a new edge from this vertex to the one chosen with equal probabilities among the rest of the vertices in the graph. Furthermore, any edge in the graph is deleted with an intensity μ. This means that the lifetime of any edge is exponentially distributed with the mean value $1/\mu$. All the processes of appending and deleting are independent.

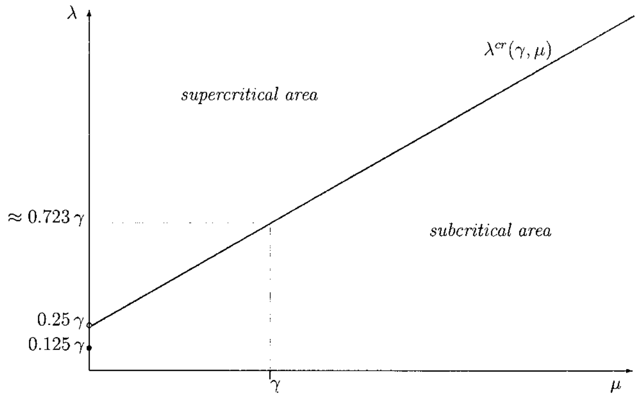

FIG. 1. Phase diagram. A graph of function $\lambda^{\rm cr}(\gamma,\mu)$, $\mu\geq 0$, for a fixed value $\gamma>0$. If the parameters belong to the supercritical area, then the corresponding graph has asymptotically almost surely the giant component of the order of the entire graph. If the parameters lie in the subcritical area, then the corresponding graphs do not possess such a component. When $\mu=0$ the critical value is $\lambda^{\rm cr}(\gamma,0)=\gamma/8$, and $\lim_{\mu\downarrow 0}\lambda^{\rm cr}(\gamma,\mu)=\gamma/4$. As $\mu\to\infty$ the graph's asymptotic line is $(\mu-\lambda)/2$. Observe that the function $\lambda^{\rm cr}(\gamma,\mu)$ is nonlinear in μ although the graph looks like a straight line for this particular choice of γ.

This model is a subgraph of a certain random grammar introduced and studied by Malyshev in Ref. [1]. More detailed analysis of the graph $(V(t),\mathcal{L}(t))$ can be found in Ref. [10], where a number of important characteristics of this graph are presented, in particular, the asymptotics of the conditional probabilities of the edges is derived.

B. Comparison and classification of the dynamical graph models

If the edges are not being deleted in our model, it clearly resembles a model of random growth. We prove below (Sec. II) that when the rate of deleting μ is zero, our model is indeed a generalized randomly grown network in a continuous time, but when $\mu\to\infty$ our model behaves as a random graph. Observe also that when μ is positive and finite our model captures the properties of the social network [6] as well, namely, that the average degree of the vertices is uniformly bounded and the system loses old enough connections. Furthermore, parameter γ allows one to choose a proper scaling of a network's growth with respect to the appearance of new connections. (A lack of this property in the known growing networks was discussed in Ref. [6].)

Clearly, when μ and γ are fixed the connectivity of our model increases with λ. Our main result here is the phase diagram on the entire space of the parameters $\gamma>0$, $\mu\geq 0$, and $\lambda>0$ (see Fig. 1 and the details in Sec. IV below). For any fixed $\gamma>0$ we find a function $\lambda^{\rm cr}(\gamma,\mu)$, $\mu\geq 0$, defined as a line separating two areas of parameters: (I) if $\lambda>\lambda^{\rm cr}(\gamma,\mu)$ then asymptotically almost every graph possesses a giant component that has a positive fraction of all the vertices, and (II) if $\lambda<\lambda^{\rm cr}(\gamma,\mu)$ the graphs do not have such a component. We will show that $\lambda^{\rm cr}(\gamma,\mu)$ for any fixed $\gamma>0$ is a continuous function in $\mu>0$, but as one can guess, this function has a jump at $\mu=0$ due to the fact that at this point the properties of the graph change drastically. Surprisingly enough, the result is that this jump is not as big as one might expect by switching from a graph that accumulates all the edges to the one that has only the relatively new edges. More precisely, we prove below that

$$\lambda^{\rm cr}(\gamma,0)=\frac{\gamma}{8} \quad \text{and} \quad \lim_{\mu\downarrow 0}\lambda^{\rm cr}(\gamma,\mu)=\frac{\gamma}{4}. \tag{2}$$

It is shown in Ref. [10] that in the supercritical phase the long paths can be composed entirely of the edges whose age is uniformly bounded. This, as well as a relatively small jump at zero confirmed by Eq. (2), implies that the process of removal of links regarded usually in a negative sense as a "permanent random damage" to a network (e.g., Ref. [4]), may have in fact a positive effect on the efficiency of the network. Namely, there is no need to preserve all the connections in order to maintain the giant component, if the system shows a sufficient enough growth. Therefore we come up with a different interpretation of the rate of removal μ: we call the value $1/\mu$ *a parameter of memory* since it shows for how long the system keeps or "remembers" any particular connection between the vertices. This leads to the following classification of the general class of the dynamical graphs with respect to the parameter of memory (or aging): dynamical graphs without memory $(1/\mu=0)\to$random graphs, dynamical graphs with an infinite memory $(1/\mu=\infty)\to$randomly grown networks, and the model we study here we shall call eventually *dynamical graphs with a finite memory* $(0<1/\mu<\infty)$.

Rephrasing the authors of Ref. [3] one may note that the randomly grown graphs have an excess of memory to be really random.

II. DEGREE DISTRIBUTION

We shall show here that the distribution of vertex degree drastically changes from being exponential when $\mu=0$ to being generalized Poisson when $\mu>0$, and finally converging to Poisson when $\mu\to\infty$.

Let $d_k(t)$ for any $k\geq 0$ denote the expected number of the vertices of degree k at time t in our graph $\mathcal{G}(t)=(V(t),\mathcal{L}(t))$. Thus we have $d_0(0)=1$ and $d_k(0)=0$ for all $k\geq 1$. Within a small time interval $(t,t+\Delta]$ the expected number of isolated vertices changes as follows:

$$d_0(t+\Delta)\approx d_0(t)-2\lambda\Delta d_0(t)+\mu\Delta d_1(t)+\gamma\Delta E|V(t)| + o(\Delta), \tag{3}$$

where $\gamma\Delta E|V(t)|$ is simply the expected number of new vertices appearing within the period $(t,t+\Delta]$, and the coefficient 2λ in the second term is due to the fact that every vertex increases its degree by producing itself an edge with probability $\lambda\Delta$, or by being chosen by any other vertex in the set $V(t)$ to be connected to. Also, every vertex decreases its degree as soon as one of the adjacent edges is deleted. Making use of the properties of the Poisson process and taking into account Eq. (1) we derive from Eq. (3):

$$d_0'(t) = -2\lambda d_0(t) + \mu d_1(t) + \gamma e^{\gamma t}. \tag{4}$$

A similar argument leads to the following equations:

$$d_k'(t) = -2\lambda d_k(t) + \mu(k+1) d_{k+1}(t) - \mu k d_k(t) + 2\lambda d_{k+1}(t) \tag{5}$$

for all $k>1$. It is natural to search for a solution of this system assuming that

$$\lim_{t\to\infty} \frac{d_k(t)}{e^{\gamma t}} = p_k, \quad k \geqslant 0. \tag{6}$$

Clearly, p_k is the probability that in the limiting graph (as $t \to \infty$) a randomly chosen vertex has a degree k, and in particular

$$\sum_{k=0}^{\infty} p_k = 1. \tag{7}$$

Then we have by Eqs. (4), (5), and (6)

$$\gamma p_0 = -2\lambda p_0 + \gamma + \mu p_1,$$

$$\gamma p_k = 2\lambda (p_{k-1} - p_k) - \mu [k p_k - (k+1) p_{k+1}], \quad k>0. \tag{8}$$

To avoid confusion we shall write sometimes $p_k = p_k(\mu, \gamma, \lambda)$. Consider some particular cases.

If $\mu = 0$ it is easy to derive from Eqs. (8) and (7)

$$p_k = p_k(0,\gamma,\lambda) = \frac{(2\lambda/\gamma)^k}{(1+2\lambda/\gamma)^{k+1}}, \quad k \geqslant 0 \tag{9}$$

for all $\gamma>0$, $\lambda>0$. Thus under the condition $\mu=0$ the degree distribution in our model follows the same exponential law as does the model of randomly grown network. In fact, when $\lambda/\gamma \leqslant 1$ we obtain exactly the same distribution as for a randomly grown network [3] with parameter $\delta := \lambda/\gamma$. On the other hand, unlike a randomly grown network, our model is defined for all values $\lambda/\gamma > 1$ as well. This is due to the fact that we have an extra parameter of the system's growth in time. Furthermore, our network evolves in a continuous time. Therefore we call this case a generalized randomly grown network.

Let now $\mu>0$. First of all we observe that when $\mu=\gamma>0$ the system (8) becomes

$$1 - \frac{2\lambda}{\gamma} p_0 + (p_1 - p_0) = 0,$$

$$\frac{2\lambda}{\gamma}(p_k - p_{k-1}) - (k+1)(p_{k+1} - p_k) = 0, \quad k>0. \tag{10}$$

Setting $\delta_k := p_k - p_{k-1}$, $k>0$, helps us to solve this system and get

$$\delta_1 = \frac{2\lambda}{\gamma} p_0 - 1,$$

and for $k>1$,

$$\delta_k = \left(\frac{2\lambda}{\gamma}\right)^{k-1} \frac{1}{k!}(p_1 - p_0) = \left(\frac{2\lambda}{\gamma}\right)^{k-1} \frac{1}{k!} \delta_1,$$

which yields

$$p_k = p_0 + \sum_{n=1}^{k} \delta_k = p_0 + \left(\frac{2\lambda}{\gamma} p_0 - 1\right) \sum_{n=1}^{k} \left(\frac{2\lambda}{\gamma}\right)^{n-1} \frac{1}{n!}. \tag{11}$$

Observing that $p_k \to 0$ as $k \to \infty$ [see Eq. (7)], we get from here

$$p_0 = \left[\left(\frac{2\lambda}{\gamma}\right)^{-1} - p_0\right](e^{2\lambda/\gamma} - 1).$$

This gives us

$$p_0 = \frac{\gamma}{2\lambda}(1 - e^{-2\lambda/\gamma}),$$

which together with Eq. (11) results in the following generalized Poisson distribution:

$$p_k = p_k(\gamma,\gamma,\lambda) = \frac{\gamma}{2\lambda} e^{-2\lambda/\gamma} \sum_{n=k+1}^{\infty} \left(\frac{2\lambda}{\gamma}\right)^n \frac{1}{n!}, \quad k \geqslant 0. \tag{12}$$

In the general case $\mu>0$ one can write the solution to the system (8) in the form

$$p_k(\mu,\gamma,\lambda) = p_k(\gamma,\gamma,\lambda) + \sum_{n=1}^{\infty} (\gamma - \mu)^n p_{kn},$$

where the coefficients p_{kn} satisfy certain recurrent relations to be derived from Eq. (8). However, we shall not focus on finding exact solutions in the general case.

Consider now another special case assuming $\lambda = \lambda(\mu)$ so that

$$\lim_{\mu\to\infty} \frac{\lambda(\mu)}{\mu} = c \tag{13}$$

for some positive constant c. In this case it is straightforward to check that in the limit when $\mu \to \infty$ while $\gamma>0$ and $c>0$ are being fixed arbitrarily, the following Poisson distribution satisfies Eq. (8):

$$\lim_{\mu\to\infty} p_k(\mu,\gamma,\lambda(\mu)) = \frac{(2c)^k}{k!}, \quad k \geqslant 0. \tag{14}$$

This is exactly the limiting (as $n \to \infty$) degree distribution for the well-known model $G_{n,p}$ which is a graph on n vertices, whose edges are independent and a probability of any edge is $p = 2c/n$ (see Ref. [7]).

Notice that although the degree distribution is not the ultimate characteristic of the graphs, we have extra information about the topology of the graphs, namely, all the models we consider here allow edges between any pair of the vertices. Thus we showed that our graph model interpolates indeed between the randomly grown graphs and the random graphs.

III. THE CASE $\mu=0$: A RANDOMLY GROWN GRAPH

We showed already that the degree distribution for our model with $\mu=0$ follows the exponential law as does the randomly grown graph [3]. But since our range of parameters is wider we shall consider also the dynamics of the sizes of the connected components.

Let $\mathcal{N}_k(t)$ denote the number of the components of size $k\geqslant 1$, and let $N_k(t)=E\mathcal{N}_k(t)$. Observe that

$$N_1(t)=d_0(t), \tag{15}$$

while for any $k>1$ we derive

$$\begin{aligned}N_k(t+\Delta)-N_k(t)=E\Bigg(&-2k\mathcal{N}_k(t)\lambda\Delta\frac{|V(t)|-k\mathcal{N}_k(t)}{|V(t)|-1}\\&-2k\mathcal{N}_k(t)\lambda\Delta\frac{k\mathcal{N}_k(t)-1}{|V(t)|-1}\\&+\lambda\Delta\sum_{n=1}^{k}n\mathcal{N}_n(t)\frac{(k-n)\mathcal{N}_{k-n}(t)}{|V(t)|-1}\Bigg)\\&+o(\Delta),\end{aligned} \tag{16}$$

where the first term on the right-hand side corresponds to the events that every vertex from any k component might get a link with a vertex that does not belong to any k component, the second term corresponds to the events that every vertex from any k component might get a link with a vertex that also belongs to a k component, and the third term corresponds to an emerging of a k component whenever two components whose sizes are summed up to k, are connected. Making use of the properties of the Poisson process we derive from here

$$N_k'(t)=-2k\lambda N_k(t)+\lambda\sum_{n=1}^{k}n(k-n)E\frac{\mathcal{N}_n(t)\mathcal{N}_{k-n}(t)}{|V(t)|-1}. \tag{17}$$

If we assume (see also Ref. [3]) that

$$\lim_{t\to\infty}E\frac{\mathcal{N}_k(t)}{|V(t)|}=\lim_{t\to\infty}\frac{N_k(t)}{e^{\gamma t}}=a_k,\quad k\geqslant 1, \tag{18}$$

then according to Eqs. (15) and (9) we have

$$a_1=\frac{1}{1+2\lambda/\gamma}, \tag{19}$$

and from Eq. (17) we get the recurrent relations

$$\gamma a_k=-2k\lambda a_k+\lambda\sum_{n=1}^{k}n(k-n)a_n a_{k-n},\quad k\geqslant 2. \tag{20}$$

Thus we come up with exactly the same relations as obtained in Ref. [3] but with the parameter λ/γ replacing δ in Eq. [3]. Recall that according to Eq. [3] the value $\delta=1/8$ is the point of phase transition. Hence, we readily get repeating the same argument as in Eq. [3] that for any $\gamma>0$ the critical value of λ at which the mean value of the largest component exhibits a phase transition is

$$\lambda^{\mathrm{cr}}(\gamma,0)=\frac{\gamma}{8}. \tag{21}$$

It was conjectured in Ref. [3] that this phase transition is of an infinite order.

IV. THE PHASE DIAGRAM

Let us fix $\gamma>0$ and $\mu\geqslant 0$ arbitrarily and consider the corresponding models for different positive values λ. Clearly, the connectivity of the model increases with λ. We already established in Eq. (21) that $\lambda^{\mathrm{cr}}(\gamma,0)=\gamma/8$. When $\mu>0$ one also expects, by analogy with classical random graphs, the existence of the unique critical value $\lambda^{\mathrm{cr}}(\gamma,\mu)$ that separates the area of parameters where asymptotically almost every graph possesses a giant connected component containing a positive fraction of all the vertices, and the area of parameters where the graphs do not have such component. Notice that the larger μ is, the shorter is the life expectation of any edge, which results in the decrease of the connectivity of the graph. This in turn implies that for every fixed $\gamma>0$ function $\lambda^{\mathrm{cr}}(\gamma,\mu)$ should be an increasing function of μ.

Let us formulate now our main result, which is an exact formula for the line of phase transition on the state space of the parameters: $\gamma>0$, $\mu>0$, $\lambda>0$. In the sequel for any two real numbers a and b we shall write $\max\{a,b\}=a\vee b$ and $\min\{a,b\}=a\wedge b$.

Theorem 4.1. For any $\gamma>0$ and $\mu>0$

$$2\lambda^{\mathrm{cr}}(\gamma,\mu)=\sup\left\{x>0:\sum_{k=2}^{\infty}x^k E\prod_{i=1}^{k-1}g(\eta_i\wedge\eta_{i+1},\gamma,\mu)<\infty\right\}, \tag{22}$$

where

$$g(t,\gamma,\mu)=\theta\left(\frac{t}{\gamma},\gamma,\mu\right)e^t=\begin{cases}\dfrac{e^{-\{\mu/\gamma-1\}t}-1}{\gamma-\mu} & \text{if } \mu\neq\gamma,\\[2ex] \dfrac{t}{\gamma}, & \text{if } \mu=\gamma,\end{cases}$$

and $\eta_1,\ldots,\eta_k$ are independent random variables with a common exponential distribution with mean value 1.

The proofs of the results of this section are given in the section below. We just briefly mention here that in the proof of Theorem 4.1 we use the ideas from branching processes to compute the probability of existence of a giant component. The main difficulty to overcome is the nonhomogeneity of the graph. We exploit in the proof the asymptotics of the probabilities of the edges derived in Ref. [10].

Next we shall discuss some properties of the phase diagram (see Fig. 1) listed in the following corollary.

Corollary 4.1. For any fixed $\gamma>0$ the function $\lambda^{cr}(\gamma,\mu)$ is a strictly increasing continuous function in $\mu>0$, such that

$$\lambda^{cr}(\gamma,0)=\frac{\gamma}{8},$$

$$\lim_{\mu\downarrow 0}\lambda^{cr}(\gamma,\mu)=\frac{\gamma}{4}, \tag{23}$$

$$\lim_{\mu\to\infty}\frac{\lambda^{cr}(\gamma,\mu)}{\mu}=\frac{1}{2}, \tag{24}$$

and for any $\mu>0$ the value $\lambda^{cr}(\gamma,\mu)$ is the smallest root of

$$1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{2x}{\mu}\right)^n\prod_{l=1}^{n}\left(\frac{1}{1+(l-1)\mu/\gamma}\right)=0. \tag{25}$$

[Note that the first equality simply repeats Eq. (21) to complete the picture here.]

It is worth noticing that the case $\mu=\gamma$ is in no way "special" compare to $\mu\neq\gamma$, and it is easy to see that the function $g(t,\gamma,\mu)$ defined in Theorem 4.1 is continuous in $\mu>0$. However, the case $\mu=\gamma$ is exactly solvable as we already saw in Eq. (12), and in particular the following formula takes place.

Remark 4.1. In the case $\mu=\gamma$ Eq. (25) becomes

$$1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(n!)^2}\left(\frac{2x}{\gamma}\right)^n=0,$$

and thus $\lambda^{cr}(\gamma,\gamma)=(a^2/8)\gamma\approx 0.723\gamma$, where a is the smallest positive real root of the Bessel function $J_0(z)$.

The results of the Corollary 4.1 show that for any fixed positive γ function $\lambda^{cr}(\gamma,\mu)$ has a jump at $\mu=0$, while this function is absolutely continuous for all $\mu>0$. This confirms in particular, a phase transition of the first order at $\mu=0$ along this parameter. We do not study here the jump of the size of the giant component of the graph at phase transition, but we expect that for any fixed $\gamma>0$ and $\mu>0$ the phase transition at λ^{cr} is of the second order by analogy with random graphs.

Observe that the result (24) could be predicted already by Eqs. (13) and (14), since $c=1/2$ is the critical value for the graph $G_{n,p}$ with $p=2c/n$.

To emphasize the role of the parameters let us write $\mathcal{L}(t)=\mathcal{L}(t,\lambda,\mu,\gamma)$. One can easily derive (see also Ref. [1]) that the mean number $E|\mathcal{L}(t)|$ of the edges satisfies the following equation:

$$\frac{d}{dt}E|\mathcal{L}(t)|=-\mu E|\mathcal{L}(t)|+\lambda E|V(t)|, \tag{26}$$

which together with Eq. (1) yields

$$E|\mathcal{L}(t,\lambda,\mu,\gamma)|=\frac{\lambda}{\gamma+\mu}(e^{\lambda t}-e^{-\mu t}). \tag{27}$$

Consider now for $\lambda^{cr}=\lambda^{cr}(\gamma,\mu)$,

$$\mathcal{R}(\mu,\gamma):=\lim_{t\to\infty}\frac{E|\mathcal{L}(t,\lambda^{cr},\mu,\gamma)|}{E|V(t)|}. \tag{28}$$

It is natural to relate this value to the efficiency of a limiting network: smaller $\mathcal{R}$ means a better efficiency since $\mathcal{R}$ defines the critical mean number of the edges that graph should possess in order to maintain the giant component of the order of the entire graph. According to Eqs. (27) and (1) we have

$$\mathcal{R}(\mu,\gamma)=\frac{\lambda^{cr}(\gamma,\mu)}{\gamma+\mu}, \tag{29}$$

which by Corollary 4.1 implies

$$\mathcal{R}(\mu,\gamma)=\begin{cases}\frac{1}{8} & \text{if } \mu=0,\\ \frac{1}{4} & \text{if } \mu\downarrow 0,\\ \frac{1}{2} & \text{if } \mu\to\infty.\end{cases}$$

Recall that the corresponding critical ratio of the mean number of the edges to the mean number of the vertices for a randomly grown graph [3] is also 1/8, while for a random graph the critical ratio is 1/2 (see Ref. [7]). This means that our model with a small positive μ needs roughly speaking, twice as many edges as does a randomly grown graph to maintain the giant component (still this is twice as less as a random graph needs to do so). However, one gains efficiency of a network by abandoning the condition of an infinite memory. Also, it is clear that unlike the scale-free networks [2] our model is robust to deletion of any $o(e^{\gamma t})$ nodes since the average degree of any vertex is bounded by a constant $2\lambda/\mu$.

V. PROOFS

A. Proof of Theorem 4.1

First of all we notice that in the case $\mu>0$ writing equations similar to Eq. (17) for the mean number of the k components does not look like a feasible task, since the mechanism of splitting a component into two parts due to removal of an edge depends on the very structure of the component. Instead, in order to study the phase diagram we shall elaborate on the ideas of branching processes proved to be useful in the random graph theory [8]. Observe that our process is nonhomogeneous. Therefore we need more delicate characteristics for the graph such as the asymptotics of the probabilities of the edges (see for the details Ref. [10]).

Consider graph $\mathcal{G}(t)$. Set $\tau_1=0$ and denote $\tau_n, n\geqslant 2$, the consecutive moments of jumps of the process $|V(t)|$, so that

$$|V(\tau_n)|-|V(\tau_n-)|=1 \quad \text{and} \quad |V(\tau_n)|=n.$$

Further we shall write

$$V(t)=\{v_0,v_{r_2},\ldots,v_{\tau_{|v(t)|}}\}, \tag{30}$$

where for each vertex v_s index s denotes the moment of appearance of this vertex in the graph. Conditionally on the set $V(t)$ we shall reveal a connected component in the graph $\mathcal{G}(t)$ using the following algorithm (see also Ref. [8]). Choose uniformly in $V(t)$ a vertex v_{s_1} to be the root. Find all the vertices connected to this vertex v_{s_1} in the graph $\mathcal{G}(t)$, denote them $\{v_{s_2^1}^1,\ldots,v_{s_2^{k(1)}}^1\}$, where $s_2^1<\cdots<s_2^{k(1)}$, and call them the first generation of the offspring of v_{s_1}. Mark v_{s_1} as *saturated*. Then for each nonsaturated but already revealed vertex v_s^1 we find all the vertices connected to v_s^1 in $\mathcal{G}(t)$ and not used previously in the algorithm; denote them v_s^2, add them to the current tree, and mark v_s^1 as saturated. We call any $v_{s'}^2$ an offspring (of the first generation) of v_s^1, and also an offspring of the second generation of the vertex v_{s_1}. We continue this process until we end up with a tree consisting of saturated vertices only. Clearly, this procedure resembles a branching process, and we shall call the offspring of v_s^{k-1} the offspring of the kth generation of v_{s_1}.

However, the number X of the offspring we assign to a given vertex v_s at each step of our algorithm depends on the set of vertices that have been used, and also on the age s of this vertex. On the condition that $T\subseteq V(t)$ is the set of the vertices in the current tree in our algorithm, let $X_k(T,s,t)$ denote the number of offspring of the kth generation of vertex v_s.

One expects that there is no long component in the graph $\mathcal{G}(t)$ if this process dies out with a probability one. On the other hand, if this process continues for a long enough time one may expect to get a positive fraction of all the vertices $V(t)$ in the current component. Thus by analogy with Theorem 5.4 [8] we have for any $\gamma>0$ and $\mu>0$,

$$\lambda^{\mathrm{cr}}(\gamma,\mu)=\sup\left\{\lambda:\sum_{k=1}^{\infty}\lim_{t\to\infty}E\;X_k(v_{s_1},s_1,t)<\infty\right\}. \quad (31)$$

Next we shall prove the following result.

Lemma 5.1. For any $k>1$

$$\lim_{t\to\infty}E\;X_k(v_{s_1},s_1,t)=(2\lambda)^k E\prod_{i=1}^{k-1}\theta\left(\frac{1}{\gamma}(\eta_i\wedge\eta_{i+1}),\gamma,\mu\right)\exp\{\eta_i\wedge\eta_{i+1}\}, \quad (32)$$

where

$$\theta(t,\gamma,\mu)=\begin{cases}\dfrac{e^{-\mu t}-e^{-\gamma t}}{\gamma-\mu} & \text{if } \mu\neq\gamma,\\ e^{-\gamma t}t & \text{if } \mu=\gamma,\end{cases} \quad (33)$$

and $\eta_1,\ldots,\eta_k$ are independent random variables with a common exponential distribution with mean value 1.

Proof. Given the set $V(t)$ [see Eq. (30)] let $p(v_s,v_u)$ denote a probability of an edge between two vertices v_s and v_u in the graph $\mathcal{G}(t)$. Then we get recursively

$$\begin{aligned}EX_k(v_{s_1},s_1,t)&=E\sum_{v_{s_1}\in V(t)}\frac{1}{|V(t)|}\sum_{v_s\in V(t)\setminus\{v_{s_1}\}}p(v_{s_1},v_s)\times X_{k-1}(\{v_{s_1},v_s\},s,t)=\cdots\\&=E\sum_{v_{s_1}\in V(t)}\frac{1}{|V(t)|}\sum_{v_{s_2}\in V(t)\setminus\{v_{s_1}\}}p(v_{s_1},v_{s_2})\cdots\\&\quad\times\sum_{v_{s_k}\in V(t)\setminus\{v_{s_1},\ldots,v_{s_{k-1}}\}}p(v_{s_{k-1}},v_{s_k}).\end{aligned} \quad (34)$$

It is trivial but worth noticing that the expectation sign here refers to the indices s_i that are the random moments of appearance of the vertices in our graph. For the rest of this proof let us write $\theta(t)=\theta(t,\gamma,\mu)$. It has been proved in Ref. [10] that on the condition that $\{v_s,v_\tau\}\in V(t)$ and given that $|V(s\vee\tau)|=\bar{V}_{s\vee\tau}$ one has the following asymptotics:

$$p(v_s,v_\tau)=2\lambda\frac{\theta(t-(s\vee\tau,t))}{\bar{V}_{s\vee\tau}}[1+\varepsilon_1(s\vee\tau,t)],$$

where $\varepsilon_1(u,t)\to0$ as $u,t\to\infty$. Consulting Ref. [10] for details one can see that Eq. (34) equals

$$(2\lambda\gamma)^k\int_0^t\cdots\int_0^t e^{-\gamma(t-s_1)}\left(\prod_{i=1}^{k-1}\exp\{-\gamma(s_i\vee s_{i+1})\}\times\theta(t-(s_i\vee s_{i+1}))e^{\gamma s_{i+1}}\right)ds_k\cdots ds_1+\varepsilon(t),$$

where $\varepsilon(t)\to0$ as $t\to\infty$. Making a change of variables in this integral we rewrite the last formula as follows:

$$\begin{aligned}&(2\lambda\gamma)^k\int_0^t\cdots\int_0^t e^{-\gamma s_1}\\&\times\left(\prod_{i=1}^{k-1}\exp\{\gamma(s_i\wedge s_{i+1})\}\theta(s_i\wedge s_{i+1})e^{-\gamma s_{i+1}}\right)ds_k\cdots ds_1\\&+\varepsilon(t)\\&=(2\lambda)^k\int_0^t\cdots\int_0^t e^{-s_1}\\&\times\left[\prod_{i=1}^{k-1}\exp\{s_i\wedge s_{i+1}\}\theta\left(\frac{s_i\wedge s_{i+1}}{\gamma}\right)e^{-s_{i+1}}\right]ds_k\cdots ds_1+\varepsilon(t).\end{aligned} \quad (35)$$

Passing now to the limit $t\to\infty$ in the last formula and taking into account Eq. (34) we obtain

$$\lim_{t\to\infty}E\;X_k(v_{s_1},s_1,t)=(2\lambda)^k E\prod_{i=1}^{k-1}\theta\left(\frac{1}{\gamma}(\eta_i\wedge\eta_{i+1}),\gamma,\mu\right)\times\exp\{\eta_i\wedge\eta_{i+1}\}, \quad (36)$$

which immediately implies Eq. (32). ■

Theorem 4.1 follows immediately by the results (32) and (31).

B. Proof of Corollary 4.1

Let

$$F_\mu(k) := E\prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}, \gamma, \mu).$$

Consider first the case $\mu > 0$. Straightforward computations yield

$$\begin{aligned} F_\mu(K) &= \sum_{n=1}^{k-2} (-1)^{n+1}\left(\frac{1}{\gamma-\mu}\right)^n \left[\prod_{l=1}^{n}\left(\frac{1}{l\mu/\gamma} - \frac{1}{1+(l-1)\mu/\gamma}\right)\right] F_\mu(k-n) \\ &= \sum_{n=1}^{k-2} (-1)^{n+1}\left(\frac{1}{\mu}\right)^n \frac{1}{n!}\left[\prod_{l=1}^{n} \frac{1}{1+(l-1)\mu/\gamma}\right] \\ &\quad \times F_\mu(k-n) =: \sum_{n=1}^{k-2} b_n F_\mu(k-n). \end{aligned} \tag{37}$$

Define now

$$\tilde{F}_\mu(x) := \sum_{k=3}^{\infty} x^k F_\mu(k) = \sum_{k=3}^{\infty} x^k \sum_{n=1}^{k-2} b_n F_\mu(k-n). \tag{38}$$

It is easy to derive that

$$\tilde{F}_\mu(x) = [F_\mu(x) + F_\mu(2)] \sum_{n=1}^{\infty} b_n x^n, \tag{39}$$

which implies

$$\tilde{F}_\mu(x) = F_\mu(2)\,\frac{\sum_{n=1}^{\infty} b_n x^n}{1 - \sum_{m=1}^{\infty} b_n x^n}, \tag{40}$$

whenever $\Sigma_{n=1}^{\infty} b_n x^n < 1$.

Notice that condition (22) is equivalent to

$$2\lambda^{\rm cr}(\gamma,\mu) = \sup\{x>0 : \tilde{F}_\mu(x) < \infty\}. \tag{41}$$

Hence, it follows immediately by Eqs. (39) and (40) that $2\lambda^{\rm cr}(\gamma,\mu)$ is the smallest positive root of the equation

$$1 = \sum_{n=1}^{\infty} b_n x^n.$$

This proves Eq. (25) when we recall the definition of the coefficients b_n from Eq. (37).

Next we shall prove Eq. (23). Observe that for any $k \geqslant 2$ the multiple integral

$$\begin{aligned} F_0(k) &:= E\prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}, \gamma, 0) \\ &= \left(\frac{1}{\gamma}\right)^k \int_0^\infty \cdots \int_0^\infty (e^{x_1 \wedge x_2} - 1)\cdots(e^{x_{k-1}\wedge x_k} - 1) \\ &\quad \times e^{-x_1 - \cdots - x_k} dx_k \cdots dx_1 =: \left(\frac{1}{\gamma}\right)^k I_k \end{aligned}$$

converges absolutely. Hence, we have by Eq. (41)

$$\lim_{\mu\to 0} 2\lambda^{\rm cr}(\gamma,\mu) = \sup\{x>0 : \tilde{F}_0(x) < \infty\}, \tag{42}$$

where

$$\tilde{F}_0(x) = \sum_{k=3}^{\infty} x^k F_0(k) = \sum_{k=3}^{\infty} \left(\frac{x}{\gamma}\right)^k I_k. \tag{43}$$

Straightforward computations yield for all $k \geqslant 3$,

$$\begin{aligned} I_k &= \int_0^\infty \cdots \int_0^\infty e^{-x_1}\left(\prod_{i=1}^{k-2} (e^{x_i \wedge x_{i+1}} - 1) e^{-x_{i+1}}\right) \\ &\quad \times x_{k-1} dx_{k-1} \cdots dx_1 \\ &= \int_0^\infty \cdots \int_0^\infty e^{-x_1}\left(\prod_{i=1}^{k-3} (e^{x_i \wedge x_{i+1}} - 1) e^{-x_{i+1}}\right) \\ &\quad \times \phi(x_{k-2}) dx_{k-2} \cdots dx_1 \end{aligned} \tag{44}$$

where ϕ is a linear operator on the space of polynomials on $\mathbf{R}_+$, such that for any $n \geqslant 1$,

$$\phi(x^n) = \frac{x^{n+1}}{n+1} + n!\,x. \tag{45}$$

Set $\phi^{(0)}(x) = x$, $\phi^{(1)}(x) = \phi(x)$, and define recursively

$$\phi^{(m+1)}(x) = \phi^{(m)}[\phi(x)], \quad m \geqslant 1.$$

Then we derive from Eq. (44)

$$I_k = \int_0^\infty e^{-x_1} \phi^{(k)}(x_1) dx_1. \tag{46}$$

Next we observe that for all $x \geqslant 0$,

$$\phi^{(2)}(x) = \phi\left(\frac{x^2}{2} + 2x\right) = \frac{x^3}{3!} + x + \phi(x).$$

Furthermore, it is trivial to check using an induction argument that in fact for all $n \geqslant 2$ and $x \geqslant 0$,

$$\phi^{(n)}(x) = \frac{x^{n+1}}{(n+1)!} + x + \phi(x) + \cdots + \phi^{(n-1)}(x). \tag{47}$$

Indeed, assume that for some $k > 2$ the relation (47) holds for all $2 \leqslant n \leqslant k$. Then we have by the linearity of the operator ϕ,

$$\phi^{(k+1)}(x)=\phi\left(\frac{x^{k+1}}{(k+1)!}+x+\phi(x)+\cdots+\phi^{(k-1)}(x)\right)$$
$$=\frac{1}{(k+1)!}\phi(x^{k+1})+\phi(x)+\cdots+\phi^{(k)}(x),$$

which by the definition (45) immediately implies Eq. (47).

Substituting Eq. (47) into Eq. (46) we obtain for all $k\geqslant 3$,

$$I_k=\int_0^\infty e^{-x}\phi^{(k-2)}(x)dx$$
$$=\int_0^\infty e^{-x}\frac{x^{n+1}}{(n+1)!}dx+\sum_{i=0}^{k-3}\int_0^\infty e^{-x}\phi^{(i)}(x)dx$$
$$=1+\sum_{i=0}^{k-3}I_{i+2}=1+\sum_{i=2}^{k-1}I_i,$$

where $I_2=1$, which gives us

$$\tilde{F}_0(x)=\sum_{k=3}^\infty\left(\frac{x}{\gamma}\right)^k I_k=\sum_{k=3}^\infty\left(\frac{x}{\gamma}\right)^k\left(1+\sum_{i=2}^{k-1}I_i\right)$$
$$=\sum_{k=3}^\infty\left(\frac{x}{\gamma}\right)^k+\sum_{i=2}^\infty\left(\frac{x}{\gamma}\right)^i I_i\sum_{k=i+1}^\infty\left(\frac{x}{\gamma}\right)^{(k-i)}$$
$$=\sum_{k=3}^\infty\left(\frac{x}{\gamma}\right)^k+\left(\tilde{F}_0(x)+\frac{x}{\gamma}\right)\sum_{k=1}^\infty\left(\frac{x}{\gamma}\right)^k. \qquad (48)$$

From here we readily derive

$$\tilde{F}_0(x)=\frac{\left(\frac{x}{\gamma}\right)^2+2\sum_{k=3}^\infty\left(\frac{x}{\gamma}\right)^k}{1-\sum_{k=1}^\infty\left(\frac{x}{\gamma}\right)^k}<\infty \quad \text{if} \quad \sum_{k=1}^\infty\left(\frac{x}{\gamma}\right)^k<1, \qquad (49)$$

i.e., as long as $x<\gamma/2$. This together with Eqs. (42) and (48) proves Eq. (23).

Finally, to prove Eq. (24) consider

$$E\prod_{i=1}^{k-1}g(\eta_i\wedge\eta_{i+1},\gamma,\mu)$$
$$=\left(\frac{1}{\mu-\gamma}\right)^k\int_0^\infty\cdots\int_0^\infty e^{-x_1-\cdots-x_k}$$
$$\times\prod_{i=1}^{k-1}(1-e^{-(\mu/\gamma-1)x_i\wedge x_{i+1}})dx_1\cdots dx_k.$$

Clearly, the integral in the last formula converges monotonously to 1 as $\mu\to\infty$. Hence by Eq. (22) we readily get

$$\lim_{\mu\to\infty}\frac{2\lambda^{\mathrm{cr}}(\mu,\gamma)}{\mu-\gamma}=1,$$

which proves Eq. (24). ■

VI. CONCLUSIONS

We have analyzed a general model of random dynamical graphs, which interpolates between randomly grown networks [3] and random graphs. This approach provides a unified point of view for these two models by placing them in one general class of dynamical graphs. The parameter of memory introduced here allows one to see clearly the similarities and differences of the models. This should be helpful for the future design of the dynamical networks. In particular, our model with a positive finite memory shares the following properties of the finite social networks [6]: the uniform boundedness of the degrees of the vertices and the decay of the old connections.

We described here the phase diagram for our model, which reveals the concurrent roles of growth and aging in the network. We showed that the critical value of the connectivity parameter λ^{cr} is a continuous function of the removal rate $\mu>0$. Our conjecture is that this line is a convex function. This would imply that $\mathcal{R}(\mu,\gamma)$ defined in Eq. (28) reaches its minimum for some $\mu_0>0$, which provides a parameter for the most efficient network in this class. We also derived here that λ^{cr} has a jump at $\mu=0$. This confirms a phase transition of the first order at $\mu=0$ along this parameter.

Another question we leave open here is the jump of the size of the largest component at the critical value of the parameter of connectivity. We expect this phase transition to be of the second order for $\mu>0$ by analogy with random graphs.

We used only the analytical methods in our study. We partially answered the questions raised about the phase diagram by the authors of Ref. [6]. A challenging task for the future study is to describe the self-organizing behavior of the dynamical graphs where the degree of a vertex depends on the history of the vertex itself. A related static model of percolation on a triangle lattice was treated analytically in Refs. [11,12]. But for a dynamical model only computational results for a finite graph are available at present (e.g., Ref. [6]).

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